

## PERIODIC MODULES OVER GORENSTEIN LOCAL RINGS

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ABSTRACT. It is proved that the minimal free resolution of a module  $M$  over a Gorenstein local ring  $R$  is eventually periodic if, and only if, the class of  $M$  is torsion in a certain  $\mathbb{Z}[t^{\pm 1}]$ -module associated to  $R$ . This module, denoted  $J(R)$ , is the free  $\mathbb{Z}[t^{\pm 1}]$ -module on the isomorphism classes of finitely generated  $R$ -modules modulo relations reminiscent of those defining the Grothendieck group of  $R$ . The main result is a structure theorem for  $J(R)$  when  $R$  is a complete Gorenstein local ring; the link between periodicity and torsion stated above is a corollary.

## 1. INTRODUCTION

This work makes a contribution to the study of eventually periodic modules over (commutative, Noetherian) local rings. In 1990, Avramov [1] posed the problem of characterizing rings that have a periodic module. Eisenbud [4] had previously shown that every complete intersection ring has a periodic module, but the question remained unanswered for other rings, even those which are Gorenstein. In Corollary 5.8, we prove that a complete Gorenstein local ring  $R$  has a periodic module if and only if there is torsion in a certain  $\mathbb{Z}[t^{\pm 1}]$ -module associated to  $R$ , where  $\mathbb{Z}[t^{\pm 1}]$  denotes the ring of Laurent polynomials. This module, which we denote  $J(R)$ , is the free  $\mathbb{Z}[t^{\pm 1}]$ -module on the isomorphism classes of finitely generated  $R$ -modules modulo relations reminiscent of those defining the Grothendieck group of  $R$ ; see Definition 2.1 and Proposition 2.7.

The main result of this paper is a structure theorem for  $J(R)$  when  $R$  is a Gorenstein local ring with the Krull-Remak-Schmidt property; see Theorem 4.2. As a corollary, we deduce that an  $R$ -module is eventually periodic if and only if its class in  $J(R)$  is annihilated by some non-zero element of  $\mathbb{Z}[t^{\pm 1}]$ . This leads to a characterization of hypersurface rings in terms of  $J(R)$ ; see Corollary 5.10.

This paper is motivated by work of D.R. Jordan [11], who defined the module  $J(R)$  and proved that if the class of a module in  $J(R)$  is torsion then the module has a rational Poincaré series. The converse, however, does not hold. Indeed, Jordan proved that, for an Artinian complete intersection ring  $R$  with codimension at least two, the class of its residue field is not torsion in  $J(R)$ . Corollary 5.10 contains this result, since the residue field of a complete intersection ring is eventually periodic if and only if the codimension is at most one.

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2. THE MODULE  $J(R)$ 

All rings considered in this paper are commutative and Noetherian. Let  $R$  be a ring and  $\mathcal{C}(R)$  the set of isomorphism classes of finitely generated  $R$ -modules; write  $[M]$  for the class of an  $R$ -module  $M$  in  $\mathcal{C}(R)$ . When the ring is clear from context, we write  $\mathcal{C}$  instead of  $\mathcal{C}(R)$ .

**Definition 2.1.** Let  $F$  be the free  $\mathbb{Z}[t^{\pm 1}]$ -module  $\mathbb{Z}[t^{\pm 1}]^{(\mathcal{C})}$ , that is,

$$F = \bigoplus_{[M] \in \mathcal{C}} \mathbb{Z}[t^{\pm 1}][M],$$

and let  $I$  be the  $\mathbb{Z}[t^{\pm 1}]$ -submodule generated by the following elements:

- (R1)  $[M] - [M']$  for every exact sequence of finitely generated  $R$ -modules  $0 \rightarrow P \rightarrow M \rightarrow M' \rightarrow 0$  with  $P$  projective;
- (R2)  $[M] - t[M']$  for every exact sequence of finitely generated  $R$ -modules  $0 \rightarrow M' \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  projective;
- (R3)  $[M \oplus M'] - [M] - [M']$  for all finitely generated  $R$ -modules  $M$  and  $M'$ .

The main object of study in this article is the  $\mathbb{Z}[t^{\pm 1}]$ -module:

$$J(R) = F/I.$$

In the following remark, we make a few observations about the module  $J(R)$ .

*Remark 2.2.* Let  $M, M'$ , and  $P$  be finitely generated  $R$ -modules with  $P$  projective.

- (1)  $[P] = 0$  in  $J(R)$ .
- (2) If  $0 \rightarrow M \rightarrow M' \rightarrow P \rightarrow 0$  is exact, then  $[M] - [M'] = 0$  in  $J(R)$ .

Indeed, for (1), note that there is an exact sequence  $0 \rightarrow P = P \rightarrow 0$ , and so the desired result follows from (R1).

To prove (2), notice that  $M' \cong M \oplus P$  since  $P$  is projective. Then in  $J(R)$ ,  $[M'] = [M] + [P]$  by (R3). Since  $[P] = 0$  in  $J(R)$ , it follows that  $[M'] = [M]$ .

The module  $J(R)$  was defined by D.R. Jordan in [11] and called the *Grothendieck module*. In Jordan's definition, the submodule  $I$  is generated by four types of elements: the three given in Definition 2.1 as well as elements of the form  $[M] - [M']$  where  $M$  and  $M'$  are modules as in Remark 2.2.(2).

*Remark 2.3.* The *Grothendieck group*  $\mathcal{G}$  of a ring  $R$  is the free  $\mathbb{Z}$ -module  $\mathbb{Z}^{(\mathcal{C})}$  modulo the subgroup generated by the Euler relations, that is, elements of the form  $[M'] - [M] + [M'']$  for each exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of finitely generated  $R$ -modules. The *reduced Grothendieck group*  $\overline{\mathcal{G}}$  of  $R$  is the group  $\mathcal{G}$  modulo the subgroup generated by classes of modules of finite projective dimension. We note that  $\overline{\mathcal{G}} = J(R)/L$ , where  $L$  is the submodule generated by the Euler relations.

**Syzygies.** In order to discuss syzygies, we recall Schanuel's Lemma; a proof can be found in [12, Thm 4.1.A].

**Schanuel's Lemma.** *Given exact sequences of  $R$ -modules*

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K' \rightarrow P' \rightarrow M \rightarrow 0$$

*with  $P$  and  $P'$  projective, there is an isomorphism  $K \oplus P' \cong K' \oplus P$  of  $R$ -modules.  $\square$*

Let  $R$  be a ring and  $M$  an  $R$ -module. Denote by  $\Omega_R M$  any  $R$ -module that is the kernel of a homomorphism of  $R$ -modules  $P \twoheadrightarrow M$  with  $P$  a finitely generated projective. While  $\Omega_R M$  depends on the choice of  $P$ , Schanuel's Lemma shows that

$M$  determines  $\Omega_R M$  up to a projective summand. Any module isomorphic to a module  $\Omega_R M$  is called a *syzygy* of  $M$ . For any  $d > 1$ , a  $d$ th syzygy of  $M$  is a module  $\Omega_R^d M$  such that  $\Omega_R^d M = \Omega_R(\Omega_R^{d-1} M)$  for some  $(d-1)$ st syzygy of  $M$ . By Schanuel's Lemma,  $\Omega_R^d M$  is also determined by  $M$  up to a projective summand. For any  $n \geq 0$ , we write  $\Omega^n M$  when the ring is clear from context.

The syzygy gives a well-defined functor on  $J(R)$ , as shown in Lemma 2.5. The following remark will aid in this discussion.

*Remark 2.4.* If  $0 \rightarrow P \rightarrow M \rightarrow M' \rightarrow 0$  is an exact sequence of  $R$ -modules with  $P$  projective, then there is a module that is a syzygy of both  $M$  and  $M'$ .

Indeed, pick a surjective map  $G' \twoheadrightarrow M'$ , with  $G'$  a projective  $R$ -module. Consider the following diagram. Let  $X$  be the pullback of  $M \rightarrow M'$  and  $G' \rightarrow M'$ . Since  $G' \rightarrow M'$  is surjective,  $X \rightarrow M$  is also surjective. Since  $G'$  and  $P$  are projective,  $X$  is projective. Hence the kernel of  $X \rightarrow M$  is a syzygy of  $M$ ; let  $N$  be this kernel. Let  $N'$  denote the kernel of  $G' \twoheadrightarrow M'$ . Then there is a commutative diagram with exact rows as follows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & M' \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & P & \longrightarrow & X & \longrightarrow & G' \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & N & \xrightarrow{\cong} & N'
 \end{array}$$

This justifies the claim.

**Lemma 2.5.** *Assigning  $[M]$  to  $[\Omega M]$  induces a  $\mathbb{Z}[t^{\pm 1}]$ -linear map*

$$\Omega: J(R) \rightarrow J(R).$$

*Proof.* By Schanuel's Lemma, the assignment  $[M] \mapsto [\Omega M]$  gives a homomorphism

$$\tilde{\Omega}: \bigoplus_{[M] \in \mathcal{C}} \mathbb{Z}[t^{\pm 1}][M] \rightarrow J(R)$$

of  $\mathbb{Z}[t^{\pm 1}]$ -modules. It is enough to check that (R1), (R2), and (R3) from Definition 2.1 are in  $\text{Ker}(\tilde{\Omega})$ , so that  $\tilde{\Omega}$  factors through  $J(R)$ ; the induced map is  $\Omega$ .

For (R3), note that for any syzygies  $\Omega M$  of  $M$  and  $\Omega M'$  of  $M'$ , the  $R$ -module  $\Omega M \oplus \Omega M'$  is a syzygy of  $M \oplus M'$ . Since  $[\Omega M \oplus \Omega M'] = [\Omega M] + [\Omega M']$  in  $J(R)$ , one finds that  $\tilde{\Omega}([M \oplus M']) = \tilde{\Omega}([M]) \oplus \tilde{\Omega}([M'])$ .

Next, we consider (R2). Given an exact sequence of finitely generated  $R$ -modules  $0 \rightarrow M' \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  projective, we show that  $[M'] - t^{-1}[M]$  is in  $\text{Ker}(\tilde{\Omega})$ . In  $J(R)$ , one has  $\tilde{\Omega}([M']) = t^{-1}[M']$ . Since  $M'$  is a syzygy of  $M$ , we have  $[M'] = [\Omega M]$  in  $J(R)$ . As  $[\Omega M] = \tilde{\Omega}([M])$ , the  $\mathbb{Z}[t^{\pm 1}]$ -linearity of  $\tilde{\Omega}$  implies that

$$\tilde{\Omega}([M']) = t^{-1}\tilde{\Omega}([M]) = \tilde{\Omega}(t^{-1}[M]).$$

Therefore (R2) is in  $\text{Ker}(\tilde{\Omega})$ .

Finally, we verify that (R1) is in  $\text{Ker}(\tilde{\Omega})$ . Given an exact sequence of finitely generated  $R$ -modules  $0 \rightarrow P \rightarrow M \rightarrow M' \rightarrow 0$  with  $P$  projective, Remark 2.4 implies there is a module  $L$  that is a syzygy of both  $M$  and  $M'$ . Therefore  $\tilde{\Omega}([M]) = [L] = \tilde{\Omega}([M'])$ .  $\square$

For  $i > 1$ , define  $\Omega^i : J(R) \rightarrow J(R)$  by  $\Omega^i = \Omega \circ \Omega^{i-1}$ . The next remark demonstrates a relationship in  $J(R)$  between the class of a module and the classes of its syzygies.

*Remark 2.6.* Let  $M$  be a finitely generated  $R$ -module. Then  $[M] = t^n[\Omega^n M]$  in  $J(R)$  for any  $n \in \mathbb{N}$ .

Indeed, (R2) implies that  $t[\Omega M] = [M]$  in  $J(R)$ . Iterating this, one finds that  $[M] = t^n[\Omega^n M]$  for all  $n \in \mathbb{N}$ .

In the next proposition, we give an alternate description of  $J(R)$  which makes the relations in this module more transparent.

**Proposition 2.7.** *Let  $F$  be the free  $\mathbb{Z}[t^{\pm 1}]$ -module  $\mathbb{Z}[t^{\pm 1}]^{(C)}$ , and let  $L$  be the  $\mathbb{Z}[t^{\pm 1}]$ -submodule generated by the following elements:*

- (R1')  $[P]$  for every finitely generated projective  $R$ -module  $P$ ;
- (R2')  $[M] - t[\Omega M]$  for every finitely generated  $R$ -module  $M$ ;
- (R3)  $[M \oplus M'] - [M] - [M']$  for all finitely generated  $R$ -modules  $M$  and  $M'$ .

*There is an isomorphism of  $\mathbb{Z}[t^{\pm 1}]$ -modules*

$$J(R) \cong F/L.$$

*Proof.* Via a proof similar to the proof of Lemma 2.5, it can be shown that assigning  $[M]$  to  $[\Omega M]$  induces a  $\mathbb{Z}[t^{\pm 1}]$ -linear map  $\Omega : F/L \rightarrow F/L$ .

Let  $\tilde{q} : F \rightarrow J(R)$  be the quotient map. We show that (R1'), (R2'), and (R3) are in  $\text{Ker}(\tilde{q})$ , and hence  $\tilde{q}$  factors through the quotient  $F/L$  via a map  $q : F/L \rightarrow J(R)$ .

The elements given by (R1') are in  $\text{Ker}(\tilde{q})$  by Remark 2.2.(1), and those from (R2') are in  $\text{Ker}(\tilde{q})$  by Remark 2.6. The elements given by (R3) are in  $\text{Ker}(\tilde{q})$  by the definition of  $J(R)$ .

Let  $\tilde{p} : F \rightarrow F/L$  be the quotient map. We show that (R1), (R2), and (R3) are in  $\text{Ker}(\tilde{p})$ , and hence  $\tilde{p}$  factors through the quotient  $J(R)$  by a map  $p : J(R) \rightarrow F/L$ . Note that the elements given by (R3) are in  $\text{Ker}(\tilde{p})$  by the definition of  $L$ . It remains to verify that (R1) and (R2) are in  $\text{Ker}(\tilde{p})$ .

First, consider (R1). Let  $0 \rightarrow P \rightarrow M \rightarrow M' \rightarrow 0$  be an exact sequence of finitely generated  $R$ -modules with  $P$  projective. By (R2'), one has

$$[M] - [M'] = [M] - t[\Omega M']$$

in  $F/L$  for any syzygy  $\Omega M'$  of  $M'$ . Given a syzygy  $\Omega M$  of  $M$ , Remark 2.4 shows that there is a projective  $R$ -module  $G$  such that  $\Omega M \oplus G$  is a syzygy of  $M'$ . Hence  $[\Omega M'] = [\Omega M \oplus G] = [\Omega M]$  in  $F/L$ , and thus

$$[M] - [M'] = [M] - t[\Omega M] = 0$$

in  $F/L$ . Therefore (R1) is in  $\text{Ker}(\tilde{p})$ .

Finally, we show that (R2) is in  $\text{Ker}(\tilde{p})$ . Let  $0 \rightarrow M' \rightarrow P \rightarrow M \rightarrow 0$  be an exact sequence of finitely generated  $R$ -modules. Then  $M'$  is a syzygy of  $M$ , so  $t[M'] - [M] = t[\Omega M] - [M] = 0$  by (R2'). Hence (R2) is in  $\text{Ker}(\tilde{p})$ .

Note that  $p \circ q$  is the identity map on  $F/L$ ; thus  $p$  is injective. Since  $p$  is a quotient map and hence also surjective,  $p$  is an isomorphism.  $\square$

Recall that a homomorphism of rings  $\varphi : R \rightarrow S$  is *flat* if  $S$  is flat as an  $R$ -module via  $\varphi$ . A straightforward argument yields the following result.

**Lemma 2.8.** *Let  $\varphi : R \rightarrow S$  be a homomorphism of rings. When  $\varphi$  is flat, the assignment  $[M] \mapsto [S \otimes_R M]$  induces a homomorphism of  $\mathbb{Z}[t^{\pm 1}]$ -modules*

$$J(\varphi) : J(R) \rightarrow J(S). \quad \square$$

**Finite projective dimension.** In [11, Prop 3] Jordan proves the following: if  $R$  is a commutative local Noetherian ring and  $M$  a finitely generated  $R$ -module, then the projective dimension of  $M$  is finite if and only if  $[M] = 0$  in  $J(R)$ . In Proposition 2.12, we extend this result to all commutative Noetherian rings.

**Definition 2.9.** Let  $(R, \mathfrak{m}, k)$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k$ , and let  $M$  be a finitely generated  $R$ -module. Set

$$\beta_i(M) = \text{rank}_k \text{Tor}_i^R(M, k);$$

this is the  $i$ th Betti number of  $M$ . The Poincaré series of  $M$  is given by

$$P_M^R(t) = \sum_{i=0}^{\infty} \beta_i(M) t^i$$

viewed as an element in the formal power series ring  $\mathbb{Z}[[t]]$ .

Let  $\mathbb{Z}((t))$  denote the ring of formal Laurent series,  $\mathbb{Z}[[t]] \left[ \frac{1}{t} \right]$ ; we view it as a module over  $\mathbb{Z}[[t]]$ . Notice that  $\mathbb{Z}[[t]]$  is a  $\mathbb{Z}[[t]]$ -submodule of  $\mathbb{Z}((t))$ .

The following proposition is [11, Lem 1]. We include the statement here for ease of reference.

**Proposition 2.10.** *Let  $R$  be a local ring. The assignment  $[M] \mapsto P_M^R(t)$  induces a homomorphism of  $\mathbb{Z}[[t]]$ -modules*

$$\pi : J(R) \rightarrow \mathbb{Z}((t))/\mathbb{Z}[[t]]. \quad \square$$

**Definition 2.11.** An  $R$ -module  $M$  has *finite projective dimension* if an  $i$ th syzygy module  $\Omega^i M$  is projective for some  $i \geq 0$ ; in this case, we write  $\text{pd}_R M < \infty$ .

By Schanuel's Lemma, an  $i$ th syzygy module is projective if and only if every  $i$ th syzygy module is projective. Observe that if  $\Omega^i M$  is projective, then  $\Omega^j M$  is projective for all  $j \geq i$ . When  $R$  is local, an  $R$ -module  $M$  has finite projective dimension if and only if  $\beta_i(M) = 0$  for  $i \gg 0$ ; see [3, Cor 1.3.2].

The following proposition was proved in [11, Prop 3] for local rings.

**Proposition 2.12.** *Let  $R$  be a commutative Noetherian ring and  $M$  a finitely generated  $R$ -module. Then  $[M] = 0$  in  $J(R)$  if and only if the projective dimension of  $M$  is finite.*

*Proof.* If the projective dimension of  $M$  is finite, then  $[\Omega^n M] = 0$  for some  $n \in \mathbb{N}$ . Hence  $[M] = 0$  by Remark 2.6.

Suppose  $[M] = 0$  in  $J(R)$ . First, we consider the case when  $R$  is local. Using the homomorphism  $\pi$  from Proposition 2.10, one finds that  $P_R(M) \in \mathbb{Z}[[t]]$ . Hence  $P_R(M)$  is a polynomial, and it follows that  $\beta_i(M) = 0$  for  $i \gg 0$ . Thus the projective dimension of  $M$  is finite.

For a general ring  $R$ , the map  $R \rightarrow R_{\mathfrak{m}}$  is flat for each maximal ideal  $\mathfrak{m}$ . Lemma 2.8 gives a homomorphism  $J(R) \rightarrow J(R_{\mathfrak{m}})$  with  $[M] \mapsto [M_{\mathfrak{m}}]$ . Thus  $[M_{\mathfrak{m}}] = 0$  in  $J(R_{\mathfrak{m}})$ , and hence  $\text{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} < \infty$ . Hence the projective dimension of  $M$  over  $R$  is finite by [2, Thm 4.5].  $\square$

### 3. MCM MODULES OVER GORENSTEIN LOCAL RINGS

In this section, we collect known, but hard to document, properties of MCM modules over Gorenstein local rings.

For the remainder of this article, let  $R$  be a local ring with residue field  $k$  and  $M$  a finitely generated  $R$ -module. Set  $(-)^* = \text{Hom}_R(-, R)$ .

A *free cover* of  $M$  [5, Def 5.1.1] is a homomorphism  $\varphi : G \rightarrow M$  with  $G$  a free  $R$ -module such that

- (1) for any homomorphism  $g : G' \rightarrow M$  with  $G'$  free there exists a homomorphism  $f : G' \rightarrow G$  such that  $g = \varphi f$ , and
- (2) any endomorphism  $f$  of  $G$  with  $\varphi = \varphi f$  is an automorphism.

A free cover is unique up to isomorphism.

Let  $\nu_R(M)$  denote the minimal number of generators of an  $R$ -module  $M$ , i.e.,  $\nu_R(M) = \text{rank}_k(k \otimes_R M)$ .

*Remark 3.1.* Every  $R$ -module admits a free cover. A homomorphism  $\varphi : R^n \rightarrow M$  is a free cover of  $M$  if and only if  $\varphi$  is surjective and  $n = \nu_R(M)$ .

A *free envelope* of  $M$  [5, Def 6.1.1] is a homomorphism  $\varphi : M \rightarrow G$  with  $G$  a free  $R$ -module such that

- (1) for any homomorphism  $g : M \rightarrow G'$  with  $G'$  free there exists a homomorphism  $f : G \rightarrow G'$  such that  $g = f\varphi$ , and
- (2) any endomorphism  $f$  of  $G$  with  $\varphi = f\varphi$  is an automorphism of  $G$ .

A free envelope is unique up to isomorphism.

*Remark 3.2.* Every finitely generated  $R$ -module  $M$  admits a free envelope. Indeed, the homomorphism  $f = (f_1, \dots, f_n) : M \rightarrow R^n$ , where  $f_1, \dots, f_n$  is a minimal system of generators of  $M^*$ , is a free envelope of  $M$ .

The free envelope of  $M$  can also be constructed as follows. Let  $R^n \twoheadrightarrow M^*$  be the free cover of  $M^*$ . Applying  $(-)^*$  to this map, one has an injection  $M^{**} \rightarrow R^n$ . The composite map  $M \rightarrow M^{**} \rightarrow R^n$  is the free envelope of  $M$ , where  $M \rightarrow M^{**}$  is the natural biduality map.

*Remark 3.3.* For a local ring  $R$ , one can choose  $\Omega M$  so that it is unique up to isomorphism by selecting  $\Omega M = \text{Ker}(\varphi)$  for a free cover  $\varphi$  of  $M$ . Hence from this section on,  $\Omega(-)$  is well-defined, up to isomorphism, on the category of  $R$ -modules.

**Definition 3.4.** The *cosyzygy module* of  $M$  is  $\Omega_R^{-1}M = \text{Coker}(\varphi)$ , where  $\varphi$  is the free envelope of  $M$ . For  $n > 1$ , the  *$n$ th cosyzygy module* of  $M$  is

$$\Omega_R^{-n}M = \Omega_R^{-1}(\Omega_R^{-(n-1)}M).$$

In [5, Sect 8.1], the authors refer to the cosyzygy module as the *free cosyzygy module*; since this is the only cosyzygy module studied in this article, we simply call it the cosyzygy module. We note that, when the module  $M$  is torsion-free, the cosyzygy module is also called the *pushforward*; see [9].

**Maximal Cohen-Macaulay modules.** A non-zero  $R$ -module  $M$  is said to be *maximal Cohen-Macaulay* (abbreviated to MCM) if  $\text{depth}_R M = \dim R$ .

If  $R$  is Cohen-Macaulay, the set of isomorphism classes of MCM, non-free, indecomposable modules generates  $J(R)$  as a module over  $\mathbb{Z}[t^{\pm 1}]$  since [3, Prop 1.2.9] implies that each  $R$ -module has a syzygy that is either MCM or zero. If  $R$  is

Gorenstein and has the Krull-Remak-Schmidt property, one can do better:  $J(R)$  is generated over  $\mathbb{Z}$  by the isomorphism classes of MCM, non-free, indecomposable modules; see Theorem 4.2.

The ring  $R$  is *Gorenstein* if it has finite injective dimension as a module over itself. Equivalently,  $R$  is Gorenstein provided it is Cohen-Macaulay and  $\text{Ext}_R^i(M, R) = 0$  for all MCM modules  $M$  and all  $i \geq 1$ ; this equivalence can be seen from [10, Satz 2.6] and [3, Prop 3.1.10].

For the remainder of this article, we focus on Gorenstein rings. The following are well-known results on MCM modules that will be used throughout the paper; for lack of adequate references, some of the proofs are given here.

*Remark 3.5.* Let  $R$  be a Gorenstein local ring and  $M$  an MCM  $R$ -module.

- (1) Let  $N$  be an  $R$ -module. If  $d \geq \dim R$ , then  $\Omega^d N$  is MCM or zero.
- (2) The natural homomorphism  $M \rightarrow M^{**}$  is an isomorphism.
- (3) The free envelope of  $M$  is an injective homomorphism.
- (4) The modules  $\Omega M$  and  $\Omega^{-1} M$  are MCM.
- (5)  $(\Omega^{-1} M)^* \cong \Omega(M^*)$ .
- (6) If  $M$  is indecomposable, then  $\Omega M$  and  $\Omega^{-1} M$  are also indecomposable.
- (7) If  $M$  has no free summands, then the modules  $\Omega M$  and  $\Omega^{-1} M$  also have no free summands.
- (8) If  $M$  has no free summands, then  $\Omega^{-n} \Omega^n M \cong M$  for all  $n \in \mathbb{Z}$ .

Property (1) follows from the Depth Lemma [3, Prop 1.2.9]. Property (2) is proved in [15, Cor 2.3]. For (4), a proof that  $\Omega M$  is an MCM module is given in [8, Lem 1.3] and [9, Prop 1.6.(2)] shows that  $\Omega^{-1} M$  is MCM.

*Proof of (3).* The free envelope of  $M$  is the composition

$$M \rightarrow M^{**} \hookrightarrow F^*,$$

where  $F \twoheadrightarrow M^*$  is the free cover of  $M^*$ . So (3) follows from (2).

*Proof of (5).* Let  $\pi : F \rightarrow M^*$  be the free cover of  $M^*$ . Since the natural map  $M \rightarrow M^{**}$  is an isomorphism,  $\pi^* : M \rightarrow F^*$  is the free envelope of  $M$  by Remark 3.2. Thus  $\Omega^{-1} M$  is defined by an exact sequence

$$0 \longrightarrow M \xrightarrow{\pi^*} F^* \longrightarrow \Omega^{-1} M \longrightarrow 0.$$

Applying  $(-)^*$  to this sequence yields the exact sequence

$$0 \longrightarrow (\Omega^{-1} M)^* \longrightarrow F \xrightarrow{\pi} M^* \longrightarrow 0.$$

As  $\pi$  is the free cover of  $M^*$ , one gets  $\Omega(M^*) \cong (\Omega^{-1} M)^*$ .

*Proof of (6).* A proof that  $\Omega M$  is indecomposable is given in [8, Lem 1.3]. We prove that  $\Omega^{-1} M$  is indecomposable. Let  $G$  be the free envelope of  $M$ . By (3), the following sequence is exact:

$$0 \rightarrow M \rightarrow G \rightarrow \Omega^{-1} M \rightarrow 0.$$

Since  $\Omega^{-1} M$  is MCM, applying  $(-)^*$  to this sequence yields the exact sequence

$$0 \rightarrow (\Omega^{-1} M)^* \rightarrow G^* \rightarrow M^* \rightarrow 0.$$

By (5),  $(\Omega^{-1}M)^* \cong \Omega(M^*)$ . Since  $M$  is indecomposable and isomorphic to  $M^{**}$ , it follows that  $M^*$  is indecomposable. As  $M^*$  is an indecomposable MCM module,  $(\Omega^{-1}M)^*$  is indecomposable by the result for syzygies. Thus  $\Omega^{-1}M$  is also indecomposable.

*Proof of (7).* Suppose  $\Omega M \cong N \oplus R$ . Let  $G$  be the free cover of  $M$ , and let  $X$  be the pushout of  $N \oplus R \twoheadrightarrow R$  and  $N \oplus R \rightarrow G$ . Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \oplus R & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & R & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since  $M$  is MCM one has  $\text{Ext}_R^1(M, R) = 0$ , so  $X \cong M \oplus R$ . Then  $\nu_R(G) \geq \nu_R(M) + 1$  as  $G$  maps onto  $M \oplus R$ . However, this is a contradiction since  $G$  is the free cover of  $M$ . Hence  $\Omega M$  has no free summand.

Since  $M$  has no free summand,  $M^*$  has no free summand. By property (5) and the result for syzygies,  $(\Omega^{-1}M)^*$  has no free summand. Hence  $\Omega^{-1}M$  also has no free summand.

*Proof of (8).* First, note that  $M \cong \Omega(\Omega^{-1}M) \oplus F'$  for some free module  $F'$  by Schanuel's Lemma. Since  $M$  has no free summands,  $M \cong \Omega(\Omega^{-1}M)$ . Next, we show that  $\Omega^{-1}(\Omega M) \cong M$ ; the result then follows by induction on  $n$ .

Let  $\Omega M \rightarrow G$  be the free envelope of  $\Omega M$ , and let  $G' \rightarrow M$  be the free cover of  $M$ . We have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \xrightarrow{i} & G & \longrightarrow & \Omega^{-1}(\Omega M) & \longrightarrow & 0 \\ & & \parallel & & \uparrow f & & \uparrow & & \\ 0 & \longrightarrow & \Omega M & \xrightarrow{j} & G' & \longrightarrow & M & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & K & \xrightarrow{\cong} & K & & \end{array}$$

Indeed, since  $M$  is MCM there are maps  $f : G' \rightarrow G$  and  $g : G \rightarrow G'$  such that  $f \circ j = i$  and  $g \circ i = j$ . Then  $i = i \circ (f \circ g)$ , and  $f \circ g$  is an isomorphism since  $i$  is the free envelope of  $\Omega M$ . Hence  $f : G' \rightarrow G$  is surjective. Thus the map  $M \rightarrow \Omega^{-1}(\Omega M)$  is also surjective. Note that the kernels of  $G' \twoheadrightarrow G$  and  $M \twoheadrightarrow \Omega^{-1}(\Omega M)$  are isomorphic by the Snake Lemma. Since  $\Omega^{-1}(\Omega M)$  is MCM and  $K$  is free,  $M \cong \Omega^{-1}(\Omega M) \oplus K$ . Since  $M$  has no free summands,  $M \cong \Omega^{-1}(\Omega M)$ .  $\square$

**Proposition 3.6.** *Let  $R$  be a Gorenstein local ring and  $M$  a finitely generated  $R$ -module.*

- (1) *If  $M$  is an MCM module, then  $t^{-n}[\Omega^{-n}M] = [M]$  in  $J(R)$  for each  $n \in \mathbb{Z}$ .*
- (2) *There is an MCM  $R$ -module  $N$  with  $[M] = [N]$  in  $J(R)$ .*

*Proof.* Remark 2.6 showed (1) for  $n \leq 0$ . Using Remark 3.5.(3), a proof similar to that of Remark 2.6 yields the desired result.

For (2), let  $d = \dim R$ . By Remark 2.6,  $t^d[\Omega^d M] = [M]$ . By (1),

$$t^{-d}[\Omega^{-d}\Omega^d M] = [\Omega^d M].$$



Hence  $[M] = [\Omega^{-d}\Omega^d M]$ .  $\square$

#### 4. GORENSTEIN LOCAL RINGS: STRUCTURE OF $J(R)$

The main result of this section is a structure theorem for  $J(R)$  when  $R$  is a Gorenstein local ring; see Theorem 4.2.

Throughout this section  $R$  will be a Gorenstein local ring. Recall that  $\Omega^n(-)$  denotes the  $n$ th (co)syzygy module, which is well-defined up to isomorphism.

Recall that a local ring  $R$  is said to have the *Krull-Remak-Schmidt property* if the following condition holds: given an isomorphism of finitely generated  $R$ -modules

$$\bigoplus_{i=1}^m M_i \cong \bigoplus_{j=1}^n N_j$$

where  $M_i$  and  $N_j$  are indecomposable and non-zero,  $m = n$  and, after renumbering if necessary,  $M_i \cong N_i$  for each  $i$ .

Henselian local rings, and in particular complete local rings, have the Krull-Remak-Schmidt property; see [14, Thm 1.8] and [14, Cor 1.9].

*Remark 4.1.* In order to set up notation for the next theorem, we first discuss a special type of  $\mathbb{Z}[t^{\pm 1}]$ -module. For this, we view  $\mathbb{Z}[t^{\pm 1}]$  as the group algebra over  $\mathbb{Z}$  of the free group  $G = \langle t \rangle$  on a single generator  $t$ ; that is,  $G \cong (\mathbb{Z}, +)$ . Let  $X$  be a set with a  $G$ -action. Let  $\mathbb{Z}X = \mathbb{Z}^{(X)}$ , the free  $\mathbb{Z}$ -module with basis given by the elements of  $X$ , and let  $\mathbb{Z}G$  be the group algebra over  $\mathbb{Z}$  of  $G$ . Then  $\mathbb{Z}X$  is naturally a  $\mathbb{Z}G$ -module [13, Ch.III, §1].

In what follows, we let

$$\mathcal{M}(R) = \left\{ [M] \in \mathcal{C}(R) \mid \begin{array}{l} M \text{ is MCM, non-free,} \\ \text{and indecomposable} \end{array} \right\}.$$

When  $R$  is clear from context, we write  $\mathcal{M}$  for  $\mathcal{M}(R)$ .

Remark 3.5 properties (4), (6), and (7) imply that  $[\Omega M]$  and  $[\Omega^{-1}M]$  are in  $\mathcal{M}$  if  $[M] \in \mathcal{M}$ . Thus there is an action of  $G$  on  $\mathcal{M}$  with  $t[M] = [\Omega^{-1}M]$  and  $t^{-1}[M] = [\Omega M]$ . Let  $\mathcal{A} = \mathbb{Z}^{(\mathcal{M})}$  be the corresponding  $\mathbb{Z}[t^{\pm 1}]$ -module. The canonical map  $\mathcal{M} \rightarrow J(R)$  induces a  $\mathbb{Z}[t^{\pm 1}]$ -linear homomorphism:

$$\Phi : \mathcal{A} \rightarrow J(R).$$

Assume  $R$  has the Krull-Remak-Schmidt property. We define a  $\mathbb{Z}[t^{\pm 1}]$ -linear homomorphism

$$\psi : \bigoplus_{[M] \in \mathcal{C}} \mathbb{Z}[t^{\pm 1}][M] \longrightarrow \mathcal{A} \quad (4.1)$$

by setting  $\psi([M]) = \sum_{i=1}^n [M_i]$ , where each  $M_i$  is indecomposable and

$$\Omega^{-(d+1)}\Omega^{d+1}M \cong \bigoplus_{i=1}^n M_i$$

with  $d = \dim R$ . Since  $\Omega^{-(d+1)}\Omega^{d+1}M$  is either zero or MCM with no free summands,  $\sum_{i=1}^n [M_i]$  is indeed in  $\mathcal{A}$ . Since  $R$  has the Krull-Remak-Schmidt property,  $\psi$  is well-defined.

**Theorem 4.2.** *Let  $R$  be a Gorenstein local ring that has the Krull-Remak-Schmidt property. Then the  $\mathbb{Z}[t^{\pm 1}]$ -linear map*

$$\Phi : \mathcal{A} \rightarrow J(R)$$

*is an isomorphism with inverse  $\Psi$  induced by  $\psi$  described in (4.1).*

*Proof.* To show that  $\psi$  induces a  $\mathbb{Z}[t^{\pm 1}]$ -linear map  $\Psi : J(R) \rightarrow \mathcal{A}$ , it suffices to show that the elements described in (R1), (R2), and (R3) from Definition 2.1 are in the kernel of  $\psi$ .

For elements given by (R3), note that

$$\Omega^{-(d+1)}(\Omega^{d+1}(M \oplus N)) \cong \Omega^{-(d+1)}\Omega^{d+1}M \oplus \Omega^{-(d+1)}\Omega^{d+1}N.$$

Then  $\psi([M \oplus N]) = \psi([M]) + \psi([N])$ , and hence (R3) is in  $\text{Ker}(\psi)$ .

Next, we consider (R2): given an exact sequence  $0 \rightarrow M' \rightarrow P \rightarrow M \rightarrow 0$  of finitely generated  $R$ -modules with  $P$  projective, we show that  $\psi([M']) = \psi(t^{-1}[M])$ . By Schanuel's Lemma there exists a free  $R$ -module  $G$  such that  $M' \cong \Omega M \oplus G$ . Since  $\psi([G]) = 0$ , we have  $\psi([M']) = \psi([\Omega M])$  in  $J(R)$ . Next, we show that  $\psi([\Omega M]) = \psi(t^{-1}[M])$ . By Remark 3.5.(8),

$$\Omega^{-(d+1)}\Omega^{d+1}(\Omega M) \cong \Omega(\Omega^{-(d+2)}\Omega^{d+2}M) \cong \Omega(\Omega^{-(d+1)}\Omega^{d+1}M).$$

Note that  $\Omega^{-(d+1)}\Omega^{d+1}(\Omega M)$  determines  $\psi([\Omega M])$  and  $\Omega(\Omega^{-(d+1)}\Omega^{d+1}M)$  determines  $\psi(t^{-1}[M])$ . Hence  $\psi([\Omega M]) = \psi(t^{-1}[M])$ , and therefore (R2) is in  $\text{Ker}(\psi)$ .

It remains to verify that (R1) is in  $\text{Ker}(\psi)$ . Let  $0 \rightarrow P \rightarrow M \rightarrow M' \rightarrow 0$  be an exact sequence of  $R$ -modules with  $P$  projective. By Remark 2.4, there are free  $R$ -modules  $G$  and  $G'$  such that  $\Omega M \oplus G \cong \Omega M' \oplus G'$ . Since  $\psi([G]) = \psi([G']) = 0$ ,  $\psi([\Omega M]) = \psi([\Omega M'])$ . As (R2) is in  $\text{Ker}(\psi)$ , one finds that  $\psi(t^{-1}[M]) = \psi(t^{-1}[M'])$  and thus  $\psi([M]) = \psi([M'])$ . Hence (R1) is in  $\text{Ker}(\psi)$ .

Thus  $\psi$  factors through the quotient  $J(R)$  via a homomorphism  $\Psi : J(R) \rightarrow \mathcal{A}$ . Notice that  $\Psi \circ \Phi$  is the identity. Indeed, if  $M$  is an MCM module with no free summands, then  $\Omega^{-(d+1)}\Omega^{d+1}M \cong M$  by Remark 3.5.(8). Hence  $\Phi$  is injective. For each  $R$ -module  $N$ , Proposition 3.6.(2) shows that there is an MCM  $R$ -module  $M$  such that  $[M] = [N]$ . Thus  $\Phi$  is also surjective and hence an isomorphism.  $\square$

*Remark 4.3.* If  $R$  is Gorenstein, Theorem 4.2 implies that  $J(R)$  is torsion-free as an abelian group. We do not know whether this holds for a general local ring  $R$ .

In [11, Lem 8], the following result is proved for Artinian Gorenstein rings.

**Corollary 4.4.** *Let  $R$  be a Gorenstein local ring, and let  $M$  and  $N$  be finitely generated MCM  $R$ -modules. Then  $[M] = [N]$  in  $J(R)$  if and only if*

$$M \oplus R^m \cong N \oplus R^n$$

*for some  $m, n \in \mathbb{Z}_{\geq 0}$ . Thus if neither  $M$  nor  $N$  has a free summand,  $[M] = [N]$  in  $J(R)$  if and only if  $M \cong N$ .*

*Proof.* If  $M \oplus R^m \cong N \oplus R^n$  for some  $m, n \in \mathbb{Z}_{\geq 0}$ , then in  $J(R)$  we have

$$[M] = [M \oplus R^m] = [N \oplus R^n] = [N].$$

Suppose that  $[M] = [N]$  in  $J(R)$ . We may assume  $M$  and  $N$  have no free summands. We first prove the result under the assumption that  $R$  is complete with respect to the maximal ideal. Complete rings have the Krull-Remak-Schmidt property for finitely generated modules; see for example [14, Cor 1.10]. Hence

Theorem 4.2 applies. Let  $d = \dim R$ , and let  $\Psi : J(R) \rightarrow \mathcal{A}$  be the isomorphism given in Theorem 4.2. Suppose

$$M = \bigoplus_{[M_\lambda] \in \mathcal{M}} M_\lambda^{e_\lambda} \quad \text{and} \quad N = \bigoplus_{[M_\lambda] \in \mathcal{M}} M_\lambda^{f_\lambda}$$

where  $e_\lambda, f_\lambda \geq 0$ . From Remark 3.5.(7) and the definition of  $\psi$  given in (4.1), one gets an equality

$$\sum_{[M_\lambda] \in \mathcal{M}} e_\lambda [M_\lambda] = \Psi([M]) = \Psi([N]) = \sum_{[M_\lambda] \in \mathcal{M}} f_\lambda [M_\lambda].$$

Since  $\mathcal{A}$  is free on  $\mathcal{M}$ , we have  $e_\lambda = f_\lambda$  for all  $\lambda$ . Therefore  $M \cong N$  as  $R$ -modules.

Now suppose that  $R$  is any local ring with maximal ideal  $\mathfrak{m}$ . Write  $\hat{R}$  for the  $\mathfrak{m}$ -adic completion of  $R$ . If  $[M] = [N]$  in  $J(R)$ , then  $[M \otimes_R \hat{R}] = [N \otimes_R \hat{R}]$  in  $J(\hat{R})$  by Lemma 2.8.

Note that an  $R$ -module  $M$  has a free summand if and only if the evaluation map  $ev : M^* \otimes_R M \rightarrow R$ , where  $\varphi \otimes m \mapsto \varphi(m)$ , is surjective. But if this map is surjective for  $M$ , then the map  $ev \otimes_R \hat{R}$  is also surjective. So since  $M$  and  $N$  have no free summands,  $M \otimes_R \hat{R}$  and  $N \otimes_R \hat{R}$  also have no free summands.

The result for complete rings then shows that  $M \otimes_R \hat{R} \cong N \otimes_R \hat{R}$  as  $\hat{R}$ -modules, and [14, Cor 1.15] implies that  $M \cong N$ .

Note that cancellation of direct summands is valid over local rings [14, Cor 1.16]. Then  $M \oplus R^m \cong N \oplus R^n$  implies that  $M \oplus R^{m'} \cong N$  or  $M \cong N \oplus R^{n'}$ . Thus if neither  $M$  nor  $N$  has a free summand,  $M \cong N$ .  $\square$

## 5. GORENSTEIN LOCAL RINGS: TORSION IN $J(R)$

Let  $R$  be a Gorenstein local ring. The main result of this section, Theorem 5.6, is that the class of a module is torsion in  $J(R)$  if and only if the module is eventually periodic. This result does not extend verbatim to Cohen-Macaulay local rings; see Example 5.11.

In the next lemma, we give a decomposition for the special type of  $\mathbb{Z}[t^{\pm 1}]$ -modules discussed in Remark 4.1.

**Lemma 5.1.** *Let  $G = \langle t \rangle$ , and let  $X$  be a set with a  $G$ -action. Then there is an isomorphism of  $\mathbb{Z}G$ -modules*

$$\mathbb{Z}X \cong \bigoplus_{n=1}^{\infty} \left( \frac{\mathbb{Z}[t]}{(t^n - 1)} \right)^{b_n} \oplus (\mathbb{Z}G)^{b_\infty}$$

where  $b_\infty, b_n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  for all  $n$ .  $\square$

*Proof.* For any  $x \in X$ , either  $t^n x \neq x$  for all  $n \neq 0$  and the orbit of  $x$  is

$$Gx = \{t^i x : i \in \mathbb{Z}\},$$

or there is an  $n > 0$  with  $t^n x = x$  and

$$Gx = \{x, tx, t^2 x, \dots, t^{n-1} x\}.$$

Then for each  $x \in X$  either  $\mathbb{Z}Gx \cong \mathbb{Z}G$  or  $\mathbb{Z}Gx \cong \mathbb{Z}[t]/(t^n - 1)$  as  $\mathbb{Z}G$ -modules; in either case, the map assigning  $x$  to 1 induces an isomorphism. Thus the decomposition of  $X$  into orbits gives the desired isomorphism.  $\square$

Recall that the *torsion submodule* of a  $\mathbb{Z}G$ -module  $L$  is

$$T_{\mathbb{Z}G}(L) = \{u \in L : ru = 0 \text{ for some } r \in \mathbb{Z}G \setminus \{0\}\}.$$

An element  $u \in T_{\mathbb{Z}G}(L)$  is said to be a *torsion element* of  $L$ .

**Proposition 5.2.** *An element  $u \in \mathbb{Z}X$  is torsion if and only if there exists an  $n \in \mathbb{N}$  such that  $(t^n - 1)u = 0$ .*

*Proof.* Suppose  $u$  is torsion in  $\mathbb{Z}X$ . Identifying  $\mathbb{Z}X$  with the right hand side of the isomorphism in Lemma 5.1, one finds that  $u$  belongs to the submodule  $\bigoplus_{n=1}^{\infty} (L_n)^{b_n}$  of  $\mathbb{Z}X$  where

$$L_n = \frac{\mathbb{Z}[t]}{(t^n - 1)}.$$

Consider the case when  $u = v + w$  where  $v \in L_\ell$  and  $w \in L_m$  for some  $\ell, m \in \mathbb{N}$ . Then  $(t^\ell - 1)v = 0$  and  $(t^m - 1)w = w$ , and hence  $(t^{m\ell} - 1)(v + w) = 0$ , since  $(t^\ell - 1)$  and  $(t^m - 1)$  both divide  $t^{m\ell} - 1$ . By induction on the number of terms in  $u$ , there exists an  $n \in \mathbb{N}$  such that  $(t^n - 1)u = 0$ .

The reverse implication is immediate.  $\square$

In light of the preceding results, Theorem 4.2 has the following corollaries.

**Corollary 5.3.** *Let  $R$  be a Gorenstein local ring that has the Krull-Remak-Schmidt property. The following statements hold.*

- (1) *An element  $u \in J(R)$  is torsion if and only if there exists an  $n \in \mathbb{N}$  such that  $(t^n - 1)u = 0$ .*
- (2) *The  $\mathbb{Z}[t^{\pm 1}]$ -module  $J(R)$  has nonzero torsion if and only if there is a finitely generated  $R$ -module  $M$  such that  $[M]$  is torsion.*

*Proof.* For (1), note that  $G = \langle t \rangle$  acts on  $\mathcal{M}$ . By Theorem 4.2,  $J(R) \cong \mathcal{A} = \mathbb{Z}\mathcal{M}$  as  $\mathbb{Z}[t^{\pm 1}]$ -modules. The result then follows from Proposition 5.2.

To prove (2), suppose  $u$  is a nonzero torsion element of  $J(R)$ . In the notation of Lemma 5.1, there is an  $n \in \mathbb{N}$  such that  $b_n \neq 0$ . By Theorem 4.2, there are some  $[M_\alpha] \in \mathcal{M}$  that generate  $L_n$ , and thus  $[M_\alpha]$  is torsion for each  $\alpha$ .

The reverse implication is immediate.  $\square$

**Torsion in  $J(R)$ .** Let  $(R, \mathfrak{m})$  be a local ring. An  $R$ -module  $M$  is said to be *periodic* if there exists an  $n \in \mathbb{N}$  such that  $M \cong \Omega^n M$ . The module  $M$  is said to be *eventually periodic* if there exists an  $n \in \mathbb{N}$  and  $\ell \in \mathbb{Z}_{\geq 0}$  such that  $\Omega^\ell M \cong \Omega^{n+\ell} M$ . In either case, the minimal such integer  $n$  is called the *period* of  $M$ . We make some observations about torsion in  $J(R)$  and eventually periodic modules.

**Remark 5.4.** Let  $M$  be a finitely generated  $R$ -module. It is easy to see that the following statements hold.

- (1)  $[M]$  is torsion in  $J(R)$  if and only if  $[\Omega^n M]$  is torsion for some (equivalently, all)  $n \in \mathbb{N}$ .
- (2)  $M$  is eventually periodic if and only if  $\Omega^n M$  is eventually periodic for some (equivalently, all)  $n \in \mathbb{N}$ .

In what follows, we write  $\widehat{M}$  for the  $\mathfrak{m}$ -adic completion of the  $R$ -module  $M$ .

**Lemma 5.5.** *Let  $M$  be a finitely generated  $R$ -module, and let  $i, j \in \mathbb{Z}_{\geq 0}$ . Then  $\Omega_R^i M \cong \Omega_R^j M$  if and only if  $\Omega_{\widehat{R}}^i \widehat{M} \cong \Omega_{\widehat{R}}^j \widehat{M}$ . In particular,  $M$  is eventually periodic if and only if the  $\widehat{R}$ -module  $\widehat{M}$  is eventually periodic as an  $\widehat{R}$ -module.*

*Proof.* Given that  $\Omega_R^i M \cong \Omega_R^j M$ , one has

$$\Omega_{\widehat{R}}^i(\widehat{M}) \cong \widehat{\Omega_R^i M} \cong \widehat{\Omega_R^j M} \cong \Omega_{\widehat{R}}^j(\widehat{M}).$$

Suppose that  $\Omega_{\widehat{R}}^i \widehat{M} \cong \Omega_{\widehat{R}}^j \widehat{M}$ . Then we have the following isomorphisms:

$$\widehat{\Omega_R^i M} \cong \Omega_{\widehat{R}}^i(\widehat{M}) \cong \Omega_{\widehat{R}}^j(\widehat{M}) \cong \widehat{\Omega_R^j M},$$

and so  $\Omega_R^i M \cong \Omega_R^j M$  by [14, Cor 1.15].  $\square$

**Theorem 5.6.** *Let  $R$  be a Gorenstein local ring, and let  $M$  be a finitely generated  $R$ -module. Then  $[M]$  is torsion in  $J(R)$  with respect to the  $\mathbb{Z}[t^{\pm 1}]$ -action if and only if  $M$  is eventually periodic. Moreover, for any  $n \in \mathbb{N}$ , the following conditions are equivalent:*

- (1)  $(t^n - 1)[M] = 0$  in  $J(R)$ .
- (2)  $\Omega^\ell M \cong \Omega^{n+\ell} M$  for  $\ell \gg 0$ .

*Proof.* Suppose  $M$  is eventually periodic. Then there are  $i, j \in \mathbb{Z}_{\geq 0}$  with  $i \neq j$  such that  $\Omega^i M \cong \Omega^j M$ . In  $J(R)$ ,  $t^{-i}[M] = [\Omega^i M] = [\Omega^j M] = t^{-j}[M]$ , and hence  $(t^{-i} - t^{-j})[M] = 0$ .

Assume  $[M]$  is torsion in  $J(R)$ . We first show that we can reduce to the case when  $R$  is complete with respect to the maximal ideal  $\mathfrak{m}$ . Let  $\widehat{M}$  denote the  $\mathfrak{m}$ -adic completion of  $M$ . Since the canonical homomorphism  $\varphi : R \rightarrow \widehat{R}$  is flat, Lemma 2.8 implies that there is a homomorphism of  $\mathbb{Z}[t^{\pm 1}]$ -modules  $J(\varphi) : J(R) \rightarrow J(\widehat{R})$  with  $J(\varphi)([M]) = [\widehat{M}]$ . Hence  $[M]$  torsion implies that  $[\widehat{M}]$  is torsion. If the result holds for complete rings, then  $\widehat{M}$  is eventually periodic as an  $\widehat{R}$ -module. Hence Lemma 5.5 implies that  $M$  is eventually periodic as an  $R$ -module.

Assume  $R$  is complete. To show that  $M$  is eventually periodic, it is enough to show that some syzygy of  $M$  is eventually periodic. We may assume  $M$  is MCM with no free summands.

Remark 3.5.(1) implies that  $\Omega^d M = 0$  or is MCM for  $d \gg 0$ . If  $\Omega^d M = 0$ , the proof is complete. If not, then replacing  $M$  by  $\Omega^d M$  we may assume that  $M$  is MCM. If  $M = N \oplus R$ , then  $[M] = [N]$  and hence  $[M]$  is torsion in  $J(R)$  if and only if  $[N]$  is torsion. Note that  $M$  is eventually periodic if and only if  $N$  is eventually periodic, since  $\Omega M \cong \Omega N$ . Thus we may assume  $M$  has no free summands.

As  $[M]$  is torsion, Corollary 5.3.(1) implies that there is an  $n \in \mathbb{N}$  such that  $(t^n - 1)[M] = 0$  in  $J(R)$ . Proposition 3.6.(1) shows that  $[\Omega^{-n} M] = t^n[M] = [M]$ . By Remark 3.5.(4), the  $R$ -module  $\Omega^{-n} M$  is MCM, and thus Corollary 4.4 implies that  $\Omega^{-n} M \oplus F \cong M \oplus G$  for some free  $R$ -modules  $F$  and  $G$ . Then, as  $R$ -modules,  $\Omega^n(\Omega^{-n} M \oplus F) \cong \Omega^n(M \oplus G)$ , and thus  $\Omega^n \Omega^{-n} M \cong \Omega^n M$ . Since  $M$  is MCM with no free summands,  $M \cong \Omega^n \Omega^{-n} M$  by Remark 3.5.(8). Hence  $M \cong \Omega^n M$ , and therefore  $M$  is eventually periodic.

It is clear that (2) implies (1). The argument in the previous paragraph along with Lemma 5.5 shows that (1) implies (2).  $\square$

**Corollary 5.7.** *Let  $R$  be a Gorenstein local ring and  $M$  a finitely generated  $R$ -module. Then  $[M]$  is torsion in  $J(R)$  with respect to the  $\mathbb{Z}[t^{\pm 1}]$ -action if and only if  $[\widehat{M}]$  is torsion in  $J(\widehat{R})$  with respect to the  $\mathbb{Z}[t^{\pm 1}]$ -action.*

*Proof.* It follows from Lemma 2.8, and was already used in the proof of Theorem 5.6, that if  $[M]$  is torsion in  $J(R)$ , then  $[\widehat{M}]$  is torsion in  $J(\widehat{R})$ .

The reverse implication is immediate from Theorem 5.6 and Lemma 5.5.  $\square$

The following corollary gives the result announced in the abstract.

**Corollary 5.8.** *Let  $R$  be a Gorenstein local ring that has the Krull-Remak-Schmidt property. The ring  $R$  has a periodic module if and only if  $J(R)$  has nonzero torsion.*

*Proof.* Corollary 5.3.(2) and Theorem 5.6 give the desired result.  $\square$

**Corollary 5.9.** *Suppose  $M = \bigoplus_{i=1}^m M_i$  for some  $R$ -modules  $M_i$ . Then  $[M]$  is torsion in  $J(R)$  if and only if  $[M_i]$  is torsion in  $J(R)$  for all  $i$ .*

*Proof.* Assume  $[M_i]$  is torsion in  $J(R)$  for all  $i$ . For each  $i \in \{1, \dots, m\}$ , there is an  $f_i(t) \in \mathbb{Z}[t^{\pm 1}]$  such that  $f_i(t)[M_i] = 0$  in  $J(R)$ . Then  $f_1(t) \cdots f_m(t)[M] = 0$ .

Suppose  $[M]$  is torsion in  $J(R)$ . We first prove the result under the assumption that  $R$  is complete. It suffices to consider the case when each  $M_i$  is indecomposable. Since  $[M]$  is torsion in  $J(R)$ , Theorem 5.6 implies that  $M$  is eventually periodic. So there is an  $n \in \mathbb{N}$  and an  $\ell \in \mathbb{Z}_{\geq 0}$  such that  $\Omega^{n+\ell} M \cong \Omega^\ell M$ , and therefore

$$\bigoplus_{i=1}^m \Omega^{n+\ell}(M_i) \cong \bigoplus_{i=1}^m \Omega^\ell(M_i).$$

We prove that each  $[M_i]$  is torsion by using induction on  $m$ , the number of indecomposable summands of  $M$ . Suppose  $M = M_1 \oplus M_2$ . Then by the Krull-Remak-Schmidt property, either  $\Omega^{n+\ell}(M_i) \cong \Omega^\ell(M_i)$  for  $i = 1, 2$  or  $\Omega^{n+\ell}(M_1) \cong \Omega^\ell(M_2)$  and  $\Omega^{n+\ell}(M_2) \cong \Omega^\ell(M_1)$ . In the first case, it is clear that  $M_1$  and  $M_2$  are eventually periodic. In the second case, note that

$$\Omega^{2n+\ell}(M_1) \cong \Omega^{n+\ell}(M_2) \cong \Omega^\ell(M_1),$$

and hence  $M_1$  is eventually periodic. Similarly,  $M_2$  is eventually periodic. Then by Theorem 5.6,  $[M_i]$  is torsion in  $J(R)$  for  $i = 1, 2$ .

Suppose  $M = \bigoplus_{i=1}^m M_i$  and that the conclusion holds for  $s < m$ . By the Krull-Remak-Schmidt property, for each  $i$  there exists  $j$  such that  $\Omega^{n+\ell}(M_i) \cong \Omega^\ell(M_j)$ . If there is an  $i$  such that  $\Omega^{n+\ell}(M_i) \cong \Omega^\ell(M_i)$ , then the result follows from the inductive hypothesis. Without loss of generality, suppose  $\Omega^{n+\ell}(M_i) \cong \Omega^\ell(M_{i+1})$  for  $1 \leq i \leq m-1$  and  $\Omega^{n+\ell}(M_m) \cong \Omega^\ell(M_1)$ . The following isomorphisms of  $R$ -modules show that  $M_1$  is eventually periodic:

$$\Omega^{mn+\ell}(M_1) \cong \Omega^{(m-1)n+\ell}(M_2) \cong \cdots \cong \Omega^{n+\ell}(M_m) \cong \Omega^\ell(M_1).$$

Similarly  $M_i$  is eventually periodic for  $2 \leq i \leq m$ , and consequently Theorem 5.6 implies that  $[M_i]$  is torsion in  $J(R)$  for all  $i$ .

Now suppose that  $R$  is any local ring and  $[M]$  is torsion in  $J(R)$ . By Corollary 5.7,  $[\widehat{M}]$  is torsion in  $J(\widehat{R})$ . Write

$$\widehat{M} = \bigoplus_{i=1}^m \left( \bigoplus_{j=1}^{a_i} M_{ij} \right) \quad \text{with} \quad \widehat{M}_i = \bigoplus_{j=1}^{a_i} M_{ij}$$

and each  $M_{ij}$  an indecomposable  $\widehat{R}$ -module. The result for complete rings implies that  $[M_{ij}]$  is torsion in  $J(\widehat{R})$  for each  $i$  and  $j$ . Then  $[\widehat{M}_i]$  is torsion in  $J(\widehat{R})$  for all  $i$ , and so Corollary 5.7 implies that  $[M_i]$  is torsion in  $J(R)$  for all  $i$ .  $\square$

Theorem 5.6 also gives a characterization of hypersurface rings in terms of  $J(R)$ . The main result of [11, Thm 7] is that (4) implies (1) holds when  $R$  is an Artinian complete intersection.

**Corollary 5.10.** *Let  $(R, \mathfrak{m}, k)$  be a Gorenstein local ring. Then the following conditions are equivalent:*

- (1)  $R$  is a hypersurface;
- (2)  $(1 - t^2) \cdot J(R) = 0$ ;
- (3)  $J(R)$  is a torsion module;
- (4)  $[k]$  is torsion in  $J(R)$  with respect to the  $\mathbb{Z}[t^{\pm 1}]$ -action.

*Proof.* (1)  $\Rightarrow$  (2). For any module  $M$  over a hypersurface one has  $\Omega^{2+\ell} M \cong \Omega^\ell M$  for  $\ell \gg 0$ , by [4, Thm 6.1], and hence  $(1 - t^2)[M] = 0$ .

(2)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (4). These implications are immediate.

(4)  $\Rightarrow$  (1). By Theorem 5.6, the module  $k$  is eventually periodic and hence the Betti numbers of  $k$  are bounded. By [7, Cor 1],  $R$  is a hypersurface.  $\square$

The following class of examples shows that the statement of Theorem 5.6 can fail for non-Gorenstein rings.

**Example 5.11.** Let  $(R, \mathfrak{m}, k)$  be a local ring with  $\mathfrak{m}^2 = 0$  and embedding dimension  $e \geq 2$ . Note that  $R$  is Cohen-Macaulay but not Gorenstein because the rank of its socle (as a  $k$ -vector space) is  $e$ . Then  $(1 - et)J(R) = 0$ , but  $R$  has no nonzero nonfree eventually periodic module.

First, we note that  $k$  is not eventually periodic but  $[k]$  is torsion in  $J(R)$ . Indeed, the sequence

$$0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow k \rightarrow 0$$

is exact, and  $\Omega k \cong \mathfrak{m}$  as  $R$ -modules. Therefore  $\Omega k \cong k^e$ , which implies that  $k$  is not eventually periodic. On the other hand,  $t^{-1}[k] = e[k]$  in  $J(R)$ , and therefore  $(1 - et)[k] = 0$ .

Let  $M$  be a nonzero, nonfree  $R$ -module. Since  $\mathfrak{m}^2 = 0$ , we have  $\Omega M \cong k^{\beta_1(M)}$ . As  $M$  is nonzero,  $\beta_1(M) \geq 1$ . Since  $k$  is not eventually periodic, the module  $M$  is not eventually periodic. However,

$$t^{-1}[M] = [\Omega M] = \beta_1(M)[k]$$

in  $J(R)$ , and therefore

$$(1 - et)[M] = t(1 - et)\beta_1(M)[k] = 0.$$

*Remark 5.12.* Using Corollary 5.3.(1), one can determine the torsion submodule of  $J(R)$  for a Gorenstein local ring that has the Krull-Remak-Schmidt property:

$$T_{\mathbb{Z}[t^{\pm 1}]}(J(R)) = \bigcup_{n=1}^{\infty} \text{Ann}_{J(R)}(1 - t^n).$$

If  $R$  is a complete intersection, then  $T_{\mathbb{Z}[t^{\pm 1}]}(J(R)) = \text{Ann}_{J(R)}(1 - t^2)$  by Theorem 5.6, since [4, Thm 5.2] shows that a periodic module  $M$  over a complete intersection has period at most two and hence  $(1 - t^2)[M] = 0$  in  $J(R)$ . For a Gorenstein ring  $R$ , however,  $[M]$  torsion in  $J(R)$  for an  $R$ -module  $M$  need not imply that  $(1 - t^2)[M] = 0$ . Indeed, for each  $n \in \mathbb{N}$ , there exists an Artinian Gorenstein local ring with a periodic module of period  $n$ ; see [6, Ex 3.6].

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